Exact scaling in surface growth with power-law noise

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We investigate a continuum formulation of surface growth following the Kardar-Parisi-Zhang equation [Phys. Rev. Lett. **56**, 889 (1986)] with a power-law distribution of the magnitudes of regional advances. This formulation describes Zhang's ballistic-deposition model [J. Phys. (Paris) **51**, 2129 (1990)] with power-law noise and possibly recent fluid-displacement experiments. Our exact theory predicts a transition of the scaling behavior from power-law-noise domination to a Gaussian-noise regime as the power increases. An apparent contradiction with previous simulations is due to a logarithmic correction to the scaling at the transition and to anomalous-growth effects. Analogous scaling behaviors are derived for the Edwards-Wilkinson model [Proc. R. Soc. London Ser. A **381**, 17 (1982)] with power-law noise. Our results are supported by simulations.

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I. INTRODUCTION AND SUMMARY

The description of the noise-driven growth of a self-affine interface far from equilibrium is a challenging problem. Self-affinity in this context means that the width $W(L,t)$ for an initially flat surface with lateral size L at time t can be expressed in the scaling form $W(L,t) = L^\alpha f(t/L^z)$ where α and z are called the roughness and dynamical exponents, respectively [1]. The scaling function f has asymptotic properties so that

$$W(L,t) \sim \begin{cases} t^\beta & \text{for } t \ll L^z \\ L^\alpha & \text{for } t \gg L^z, \end{cases} \tag{1}$$

where $\beta = \alpha/z$ is the early-time exponent. This property occurs for many models of surface growth. An example is growth models in the universality class of the continuum equation of Kardar, Parisi, and Zhang (KPZ) [2], which should describe fluid flow, vapor deposition, and directed polymers. In 1+1 dimensions, both theory and simulations give $\alpha = 0.5$ for bounded Gaussian noise. However, experiments on immiscible fluid displacement give α in the range 0.73–0.89 [3,4]. Zhang suggested an explanation by postulating that the probability distribution of the noise, η, in the experiments has a power-law tail. The distribution can be realized as

$$P(\eta) = \begin{cases} \mu \frac{1}{\eta^{\mu+1}} & \text{for } \eta \geq 1 \\ 0 & \text{for } \eta < 1. \end{cases} \tag{2}$$

The exponents α and z are functions of μ [5]. The existence of this type of noise is supported by another recent fluid-displacement experiment [6].

In addition to its possible experimental relevance, Zhang's model is important theoretically. For example, one should expect that α and z would be universal for any given μ for this model. This is not supported by numerical simulation, and model-dependent values have been obtained [7,8]. In fact, the existence of any continuum description has been doubted [9]. Further, the self-affinity of the interface itself has been questioned and multi-self-affinity suggested [10].

In this paper, we give an exact theory of growth with power-law noise, which should clarify the situation. We show the surprising result that α is *exactly* given by the estimate of Zhang [9] and Krug [11]:

$$\alpha = \begin{cases} \alpha_p(\mu) & \text{for } d+1 \leq \mu \leq \mu_c \\ \alpha_G & \text{for } \mu_c \leq \mu \end{cases} \tag{3}$$

in $d+1$ dimensions, where

$$\alpha_p(\mu) = \frac{d+2}{\mu+1} \tag{4}$$

and α_G is the roughness exponent for the KPZ equation with Gaussian noise. The critical μ_c is defined by $\alpha_p(\mu_c) = \alpha_G$. The dynamical exponent z is given by the identity [12,13]

$$\alpha + z = 2. \tag{5}$$

This result deviates significantly from previous numerical findings at $\mu \simeq \mu_c$ both in $1+1$ [5,7,8] and $2+1$ [14] dimensions and has led to the general belief that it can only be taken as a rough estimate. The expression (4) was derived using simple scaling arguments under the assumption that exceptionally large fluctuations dominate the roughness whenever the power-law noise is relevant. The large fluctuations correspond to the tail of the distribution in Eq. (2) and scale as

$$P(b\eta) = b^{-(\mu+1)} P(\eta). \tag{6}$$

Together with the exponent identity of the underlying KPZ equation [Eq. (5)], some simple algebra gives the results [9,11]. In this work, we will make the argument concrete so that its deficiencies are analyzed. We found that Eqs. (3)–(5) are indeed exact, but there is a very long crossover. This accounts for the apparent contradictions

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with previous simulations. The principal reason for the crossover is that the scaling of the noise in Eq. (6) does not hold at small η due to the existence of a fixed lower cutoff in Eq. (2). This leads to a logarithmic correction to the scaling at $\mu = \mu_c$. Another important source of error is anomalous growth, which occurs when the surface is highly stretched so that its evolution is dominated by lattice discretization effects [15,16]. The main results in this paper have been summarized earlier [17].

Section II discusses the Edwards-Wilkinson (EW) model [18] with power-law noise. This model, while easily solvable, already exhibits the transition between power-law noise and Gaussian noise and a logarithmic correction to the scaling at the transition. We will derive the exact scaling exponents and the correction using the properties of the stable laws for sums of independent random variables [19]. This particular approach is not directly generalizable to the KPZ case.

Section III presents the scaling argument and establishes its exactness for the KPZ case. We start in Sec. III A with a continuum formulation of a regional surface-advance model. The model treats a normal KPZ surface occasionally perturbed by instantaneous advances of whole regions. (A KPZ surface is one that evolves according to the standard KPZ equation with Gaussian noise.) The magnitude of the perturbation follows a power law with an appropriate *scale-dependent* lower cutoff, enabling, by construction, perfect scaling of the noise analogous to Eq. (6). Now both the scaling of the underlying KPZ dynamics and the perturbation are exact and Eqs. (3) can be proved quite easily. This is numerically verified in Sec. III B. Much of the anomalous growth is suppressed by simulating the underlying KPZ surface with a ballistic-deposition model with partial sliding. Finally, we remove the *ad hoc* ingredients of the model. In Sec. III C, we argue that introducing a more physical fixed lower cutoff of the noise magnitude leads to crossover effects and Eqs. (3) remain exact. A logarithmic crossover at the transition is numerically verified. Section III D shows that representing the large fluctuations by regional surface advances is appropriate for the original Zhang model.

Section IV sketches how the above scaling argument is applied to the EW case, illustrating the full correspondence between the two problems. The explicit calculation of the logarithmic correction in this approach is demonstrated. We discuss that the proper representation of the power-law noise in the continuum formulation is drastically different, even though it is the same as the KPZ case in the discrete models.

Section V focuses on the effect of anomalous growth. Our scaling argument is based on an underlying KPZ equation. However, for $\mu = d+1$ so that $\alpha = 1$, Zhang's model generates a fractal aggregate and the interface is macroscopically rough [20]. Any continuum description is likely to break down. In Ref. [20], we have reported an alternative derivation of the exponents for $\mu = d+1$ by analyzing anomalous growth only. We will show that though the two approaches give the same α and z, they predict different extended scaling behaviors with respect to the noise intensity. We found that only the anomalous growth picture is the correct description for Zhang's model at $\mu = d+1$. As a result, while Zhang and Krug's scaling argument has been thought to be an approximation at larger μ and exact at $\mu \simeq d+1$, our results show surprisingly that it is exact at larger μ but inappropriate for the discrete models at $\mu \simeq d+1$. We also discuss how anomalous growth contributes to spurious effects in the surface correlation measurements, which were interpreted as multi-self-affine scaling [10]. We conclude in Sec. VI with some further discussion.

II. EDWARDS-WILKINSON MODEL WITH POWER-LAW NOISE

Edwards and Wilkinson introduced the following Langevin equation to describe sedimentation [18]:

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h(x,t) + \eta(x,t), \tag{7}$$

where $h(x,t)$ is the surface height and $\eta(x,t)$ the noise. To simulate a surface following the EW equation in 1 + 1 dimensions, we adopt the following discrete evolution rule:

$$\begin{aligned} h(x,t+1) &= h(x+1,t+1) \\ &= \tfrac{1}{2}[h(x,t) + h(x+1,t)] + \eta(x,t). \end{aligned} \tag{8}$$

We assume periodic boundary conditions and a parallel updating scheme according to which x runs through all odd (even) sites at odd (even) t. The discrete uncorrelated variables $\eta(x,t)$ follow the power-law distribution in Eq. (2). Starting from an initially flat surface, we measure the average saturated surface width $W(L)$ as a function of the lattice size L after the roughness has fully developed. For the very broad noise distribution with $\mu \leq 2$, the rms fluctuation of the surface diverges. We define $W(L)$ as the mean absolute value of the surface fluctuation. A fit to $W(L) \sim L^\alpha$ for $L = 32$ to 256 gives the roughness exponents α shown in Fig. 1 as a function of μ. The solid line is the prediction of Krug's scaling argument [11]:

$$\alpha = \begin{cases} (2+d)/\mu - d & \text{for } \mu \leq 2 \\ (2-d)/2 & \text{for } 2 \leq \mu\ . \end{cases} \tag{9}$$

The expected sharp transition at $\mu = 2$ is smoothed out in the numerical measurement. This discrepancy is strikingly similar to that observed for the KPZ case [7,14]. We will prove that the discrepancies in both cases are due to crossovers of analogous origins. From our simulations, we also noticed that the measured α decreases slowly with increasing L, indicating a long crossover. However, we are limited to rather small lattices because the dynamical exponent $z = 2$ is rather large so that the surfaces relax slowly. In addition, a broad noise distribution at small μ implies a noisy system and long runs are required to achieve reasonable statistics.

We now present a continuum theory of the EW model with power-law noise. Both the exactness of Eq. (9) and

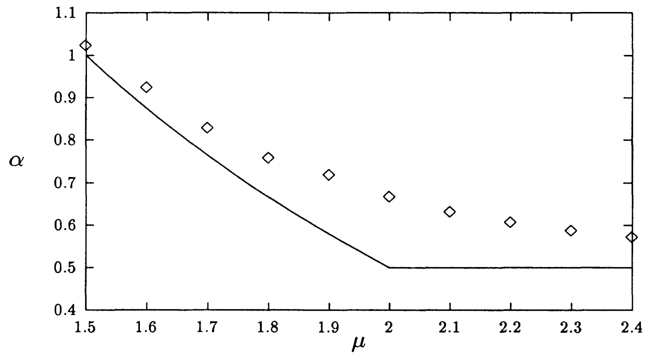

FIG. 1. Measured roughness exponent α as a function of the power of the noise distribution μ for the Edwards-Wilkinson model. The solid line plots the exact result. The discrepancy is due to a long crossover.

the crossover effects will be derived. The noise term in the EW equation (7) is realized as a superposition of independent events:

$$\eta(x,t) = \sum_i \eta_i \delta(x - x_i)\delta(t - t_i), \tag{10}$$

where η_i follows the power-law distribution (2). The distribution of the locations (x_i, t_i) of the noise events is homogeneous in both space and time, but can be either random or regular. The scaling properties of the noise can be characterized by the local fluctuation $\eta_I \Delta x^d \Delta t$ over a small region I of widths Δx and Δt, defined by

$$\eta_I \Delta x^d \Delta t = \int_I \eta(x,t)\ d^d x\, dt = \sum_{i'=1}^{N} \eta_{i'} . \tag{11}$$

The summation is restricted to the N noise events inside the region I. Due to the homogeneous distribution,

$$N \sim \Delta x^d \Delta t. \tag{12}$$

At large length scales, N is large and the sum in Eq. (11) has a probability distribution converging to a stable law. From the central limit theorem and its counterparts for distributions with diverging variances, we learn which stable law the distribution with various μ will converge to. Using the properties of the stable laws [19], we obtain

$$\sum_{i'=1}^{N} \eta_{i'} \sim \begin{cases} N^{1/\mu}\ \xi_\mu & \text{for } \mu < 2 \\ (N \ln N)^{1/2}\ \xi_G & \text{for } \mu = 2 \\ N^{1/2}\ \xi_G & \text{for } \mu > 2\ , \end{cases} \tag{13}$$

where ξ_G is a standardized Gaussian variable and ξ_μ follows the asymmetric Levy stable law parametrized by μ. It is already evident that, for $\mu < 2$, new universality classes characterized by the Levy stable law arise. For $\mu \geq 2$, the model crosses back to Gaussian-noise behavior while the crossing is marginal at $\mu = 2$.

Applying a scale transformation $x = bx'$, $t = b^z t'$ and correspondingly for Δx and Δt, the transformed η_I is deduced from Eqs. (11)–(13) to be:

$$\eta_I' \sim \begin{cases} b^{(\frac{1}{\mu}-1)(d+z)} \eta_I & \text{for } \mu < 2 \\ b^{-\frac{1}{2}(d+z)} (\ln b)^{\frac{1}{2}} \eta_I & \text{for } \mu = 2 \\ b^{\frac{1}{2}(d+z)} \eta_I & \text{for } \mu > 2\ . \end{cases} \tag{14}$$

Consider first $\mu < 2$. Assuming the scaling form $h(x,t) = b^\alpha h'(x',t')$, Eq. (7) is transformed to

$$b^{\alpha - z} \frac{\partial h'}{\partial t'} = \nu b^{\alpha - 2} \nabla^2 h' + b^{(\frac{1}{\mu}-1)(d+z)} \eta_I'(x,t), \tag{15}$$

where η is substituted by η_I, which is possible because of the linearity of the problem. The equation is invariant provided

$$\alpha - z = \alpha - 2 = \left(\frac{1}{\mu} - 1\right)(d+z). \tag{16}$$

Using this equation and a similar one for $\mu > 2$, Eq. (9) and $z = 2$ are proved. The transition at $\mu = 2$ is special. A scaling form with the logarithmic correction

$$h(x,t) = b^{\frac{1}{2}} (\ln b)^{\frac{1}{2}} h'(x',t') \tag{17}$$

is required to achieve the invariance. Equation (1) is then modified to

$$W(L,t) \sim \begin{cases} t^{\frac{1}{4}} (\ln t)^{\frac{1}{2}} & \text{for } t \ll L^z \\ L^{\frac{1}{2}} (\ln L)^{\frac{1}{2}} & \text{for } t \gg L^z. \end{cases} \tag{18}$$

The discrepancy between the values of the scaling exponents from simulations and Eq. (9) is due to the lack of convergence at finite-length scales of η_I towards the stable laws. The convergence near the transition is slower and is only marginal at the transition, corresponding to the logarithmic correction. In the simulations, generating the discrete noise $\eta(x,t)$ in Eq. (8) directly according to the stable laws should eliminate the crossovers.

III. SCALING ARGUMENT

A. Regional surface-advance model

We now focus on the nonlinear KPZ case. We first start with a simplified model, which will be generalized subsequently. Consider the continuum equation

$$\frac{\partial h}{\partial t} = \frac{\lambda}{2} (\nabla h)^2 + \eta(x,t), \tag{19}$$

where $h(x,t)$ is the interface height profile. The power-law noise $\eta(x,t)$ is a superposition of independent events:

$$\eta(x,t) = \sum_i g(\eta_i, x - x_i)\delta(t - t_i), \tag{20}$$

where g corresponds to the advance of a whole region of the surface by η_i. We assume a scaling form

$$g(\eta, x) = \eta p[(A^\sigma/\eta)^{1/\alpha} x], \tag{21}$$

where A and σ are constants to be specified later and the noise profile p is any positive-definite smooth function with bounded support whose maximum is $p(0) = 1$. The noise events have a uniform density $\overline{n}(x,t,\eta)$ in space and time and a power-law distribution in magnitude, i.e.,

$$\overline{n}(x,t,\eta) = \begin{cases} A\mu\frac{1}{\eta^{\mu+1}} & \text{for } \eta \geq \eta_m \\ 0 & \text{for } \eta < \eta_m. \end{cases} \tag{22}$$

Therefore A controls the intensity of the noise and η_m is the lower cutoff.

We apply a scale transformation $x = bx'$, $t = b^z t'$, $h(x,t) = b^\alpha h'(x',t')$, $\eta = b^\alpha \eta'$, $\eta_m = b^\alpha \eta'_m$, and $\eta(x,t) = b^{\alpha-z}\eta'(x',t')$. The model defined by Eqs. (19)–(22) is invariant if we put $\alpha + z = 2$ [i.e., Eq. (5)] and $\alpha\mu = d + z$, which imply $\alpha = \alpha_p(\mu)$. In the derivation, the transformation of Eq. (22) is obtained from the invariance of $\overline{n}(x,t,\eta)d^dx\,dt\,d\eta$. We also assume that λ has no correction upon rescaling and will be justified. Using this scale invariance, the values of any given statistical measurement at different scales can be related. In particular, the steady-state surface width is now a function of L and η_m. We have

$$W(bL, b^{\alpha_p}\eta_m) = b^{\alpha_p} W(L,\eta_m). \tag{23}$$

There exists a similar invariance property with respect to the following change of the noise intensity: $A = cA'$, $t = c^\chi t'$, $h(x,t) = c^\sigma h'(x',t')$, $\eta = c^\sigma \eta'$, and $\eta_m = c^\sigma \eta'_m$. Analogous calculations generalize Eq. (23) to

$$W \sim L^{\alpha_p} A^{\sigma_p}, \tag{24}$$

provided we put

$$\eta_m \sim L^{\alpha_p} A^{\sigma_p}, \tag{25}$$

$$\sigma = \sigma_p(\mu) = \frac{1}{\mu+1}, \tag{26}$$

and $\chi = -\sigma$. We stress that at the present stage where we scale the cutoff and the noise profile, the scaling properties are *not* approximations or asymptotic results but are *exact* for all L and A and for all $\mu \geq d+1$, even beyond μ_c.

We now add more terms to Eq. (19) to produce a generalized KPZ equation with power-law regional advance:

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \xi(x,t) + \eta(x,t), \tag{27}$$

where $\xi(x,t)$ is the uncorrelated Gaussian noise. Suppose that $\mu < \mu_c$, i.e., $\alpha_p > \alpha_G$, and that we scale the system according to the usual KPZ exponents α_G and z_G; it is easy to see that the power-law-noise intensity A increases under coarsening. Therefore at sufficiently large length scales, the surface is effectively "stretched" compared to the pure KPZ case. This stretching upsets the balance of the terms which have the same scaling behavior in the ordinary KPZ case. That is, by simple power counting, $\nu\nabla^2 h$ and ξ are irrelevant compared to $\frac{\lambda}{2}(\nabla h)^2$. This can be explicitly demonstrated in 1+1 dimensions using the renormalization flow equations in Ref. [2] [21]. They are directly applicable here because the smooth noise profile being in the long-wavelength limit is irrelevant in the renormalization of the underlying KPZ equation. The coefficient λ is not renormalized due to the invariance of the KPZ equation under an infinitesimal tilt [13]. As a result, Eq. (27) reduces to Eq. (19) and falls into the same universality class. At any finite scale, this causes a crossover. If A is small, so that the power-law noise is negligible, we expect KPZ behavior. However, upon coarse graining, so that Eq. (19) is approached, power-law noise dominates and Eq. (24) applies. The effective roughness exponent $\alpha(L)$ for system size L thus crosses over from α_G to α_p as L increases. In a similar way, for $\mu > \mu_c$, A decreases under rescaling so $\alpha(L)$ crosses over from α_p to α_G instead. This proves Eq. (3) for this case.

B. Simulations and results

We verify the above results numerically for 1+1 dimensions using a ballistic-deposition model with power-law regional surface advances. We take the basic interface evolution rule with Gaussian noise to be

$$h(x,t+1) = \tfrac{1}{2}[\max\{h(x,t), h(x+1,t), h(x-1,t)\} + h(x,t)] + \xi(x,t), \tag{28}$$

where $\xi(x,t)$ is a uniform random deviate in the range 0 to 1. The motivation for this unusual choice will be given in Sec. V. Periodic boundary conditions and a parallel updating algorithm are used so that growth occurs at all even (odd) sites at even (odd) time steps. Power-law noise occurs independently with much lower probability according to Eq. (22). If power-law noise η_i occurs at a site x_i, a whole neighborhood advances instantly from h to $h + g(\eta_i, x - x_i)$. We take a truncated parabolic profile function

$$p(y) = \max\{1 - [y/(4\times 0.015^{1/3})]^2, 0\} \tag{29}$$

and $\eta_m = (\frac{L}{16})^{\alpha_p}(\frac{A}{0.015})^{\sigma_p}$. Figure 2 shows a snapshot of a growing aggregate with $L = 128$ and $\mu = 3$. One rare noise event occurred that disturbed the surface by pushing a neighborhood forward, leaving a big void.

A log-log plot of rms surface width W against L for $\mu = 2$ to 6 for a subset of our data with $A = 0.015$ was shown in Fig. 1 of Ref. [17]. Also shown was the pure Gaussian-noise case ($A = 0$), which bounds the other curves from below. The local slopes of the curves correspond to $\alpha(L)$. We can see the two different crossovers mentioned above separated by the critical $\mu_c = 5$ case. To check Eq. (24), we have to adjust A to go beyond the crossover region to the power-law-noise-dominated regime. To do this, we need W to be significantly larger than for $A = 0$. However, if W is too large, examination of the interface reveals that microscopic roughness is smoothed out due to the strong stretching. The interface is in a model-dependent "anomalous-growth mode," which is incompatible with the continuum description [15] and resembles step flow in this case. Therefore Eq. (24) is approached only for a narrow window: see

FIG. 2. Growing aggregate of ballistic deposition with rare instantaneous regional surface advances at $\mu = 3$ and $L = 128$. The big void results from a rare event.

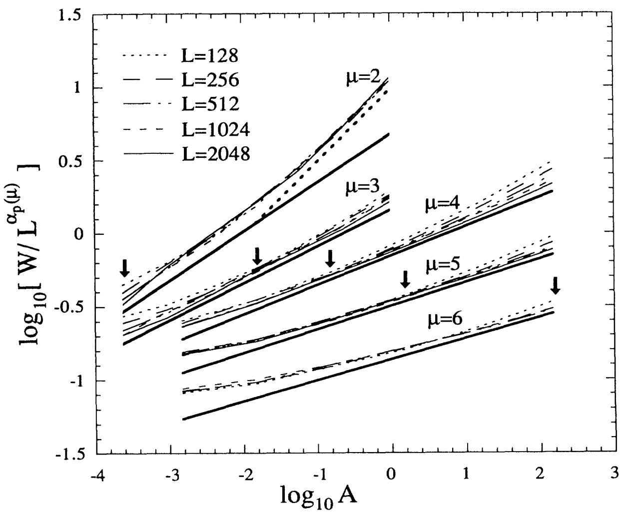

FIG. 3. Rescaled surface width $W/L^{\alpha_p(\mu)}$ against noise intensity A in log-log scale for $\mu = 2$ to 6. The vertical scale is shifted by 1.5, 0.8, 0.4, 0, and -0.4, respectively. The arrows and the solid lines are explained in the text.

Fig. 1 of Ref. [17]. At large L and intermediate values of A corresponding to this region, the fitted exponent α was shown in Fig. 2 of Ref. [17]. $\alpha = \alpha_p(\mu)$ is nicely verified for $\mu \gtrsim 3$. The discrepancy for $\mu \simeq 2$ is due to the remnant of the crossover, which is more pronounced for $\alpha_p(\mu)$ very different from α_G. The transition stated in Eqs. (3) is not evident here because we are showing results in the power-law-noise-dominated regime instead of the asymptotic regime.

The whole data set is presented in Fig. 3, where $W/L^{\alpha_p(\mu)}$ is plotted against A in log-log scale. The arrows point to the subset of data for Fig. 2 of Ref. [17], where the good-data collapse for $\mu \gtrsim 3$ corresponds to the fact that $\alpha = \alpha_p(\mu)$. The good agreement with the expected slope (thick solid lines) verifies Eq. (26). For $\mu = 3$ and 4, both the data collapse and the expected slope break down for small A due to crossover and for large A due to anomalous growth, while they hold for a narrow range of A, which widens as L increases. For the critical $\mu = 5$, data collapse is good for intermediate values of A down to 0 in spite of the crossover because $\alpha_p(5) = \alpha_G$. For $\mu = 2$, the collapse is good for large A because it happens accidentally that anomalous growth also gives $\alpha = 1$ [20].

C. Lower cutoff of noise magnitude

We have so far proved formulas (3), (5), and (26) for the generalized KPZ equation (27). We will now further generalize the results. In Zhang's model, the evolution is defined by an equation similar to Eq. (28) with ξ replaced by the power-law-distributed variable η. We seek to describe it using Eq. (27) by defining an effective lower cutoff η_{m0}. Large values of the noise, with $\eta > \eta_{m0}$, correspond to the rare large fluctuations represented by the $\eta(x,t)$ term in Eq. (27). The truncated lower part of the spectrum with $\eta < \eta_{m0}$ behaves like Gaussian noise and can be represented by the $\xi(x,t)$ term.

We are thus led to consider a constant cutoff, $\eta_m = \eta_{m0}$, instead of Eq. (25). For $\mu < \mu_c$, assume that L is large enough so that Eq. (27) reduces to Eq. (19) and hence Eq. (23) holds. Rearranging Eq. (23) gives

$$W(L, \eta_{m0}) = (L/L_0)^{\alpha_p} W(L_0, \eta_m(L)), \tag{30}$$

where $\eta_m(L) = (L_0/L)^{\alpha}\eta_{m0}$. As $L \to \infty$, $\eta_m(L) \to 0$. The term $\lim_{\eta_m \to 0} W(L_0, \eta_m)$ approaches a constant, so that Eq. (24) still holds asymptotically. This can be demonstrated explicitly for the EW model (Sec. IV). For the current nonlinear case, we argue that if W were to diverge, it could be made arbitrarily large by decreasing η_m and the roughness would be dominated by the part of the power-law-noise spectrum arbitrarily close to zero. This implies that the large fluctuations are irrelevant, which is not true in this case. For $\mu > \mu_c$, the large fluctuations are irrelevant compared to both the Gaussian noise and the smaller power-law noise and hence $\alpha = \alpha_G$. Therefore, all our scaling formulas remain intact. At the critical $\mu = \mu_c$, we suspect that W diverges logarithmically so that Eq. (30) gives

$$W(L, \eta_{m0})^2 \sim L^{2\alpha_G} \ln_{10}(L)\,. \tag{31}$$

This logarithmic correction is motivated by the similar result for the EW model derived in Sec. II and is indeed the principal reason why criticality was previously overlooked [5,7,14]. We simulated Zhang's model using Eq. (28) with ξ replaced by the power-law noise η following Eq. (2) for $\mu = 5$. A fit of the measured surface width to the form $W(L) \sim L^{\alpha}$ for $L = 128$ to 4096 gives

$\alpha \simeq 0.59$ consistent with previous results [7]. However, fitting over ranges from $\frac{L}{2}$ to $2L$ gives the L-dependent effective $\alpha(L)$ shown in Fig. 4 of Ref. [17], implying that $\alpha = \lim_{L\to\infty} \alpha(L) \ll 0.59$. The $W(L)$ curve is in fact well described by Eq. (31) as demonstrated in the inset. This supports our value $\alpha = 0.5$.

The result $\alpha \simeq 0.5$ for $\mu = 5$ has been obtained numerically using sequential updating [8]. Our simulations show that this introduces another large correction due to the larger "intrinsic width" [22]. This correction might partially cancel the one due to the cutoff but significantly increases the uncertainty in the measured α.

D. General noise profile

Finally we generalize the result to other noise profiles. Consider replacing Eq. (21) by

$$g(\eta, x) = \eta p[\eta^{-1/\gamma} x], \tag{32}$$

where $\gamma > \alpha$. Under the scale transformation, Eq. (32) is no longer invariant, as is Eq. (21), but $p(y)$ has to be replaced by $p(b^{1-\alpha/\gamma} y)$. As $b \to \infty$, $p(y)$ approaches the fixed point

$$p^*(y) = \begin{cases} 1 & \text{for } y = 0 \\ 0 & \text{for } y \neq 0, \end{cases} \tag{33}$$

which is a spike with finite height. Invariance is then restored and our scaling formulas are applicable again. Similar arguments can be applied to the general case where g is any given stochastic single-peaked positive smooth function satisfying $g(\eta, 0) \sim \eta$ with the widths of the peaks bounded by $\eta^{1/\alpha}$ as $\eta \to \infty$.

For Zhang's model, an initial large fluctuation η creates huge roughness on the surface, leading to anomalous-growth. Only after the fluctuation diffuses so that the local slope everywhere is restored to moderate values will the continuum description be valid again. The advance of the interface during this anomalous-growth period less the average growth can be set to $g(\eta, x)$. By examining growth rules like Eq. (28) and similar variants, we see that g defined in this way can easily satisfy the criteria specified above. In addition, the anomalous-growth period is negligible compared to the dynamical time scale of the interface for large scales, justifying the assumption of instantaneous regional surface advances. Therefore our theory should apply also for Zhang's model. For similar reasons, it also describes a growth model where the elementary events follow a power-law waiting-time distribution [23], and a recent simulation where disks of random radii instead of sticks are deposited [24]. Due to the very wide range of its applicability, it is possible that this theory also describes the fluid-displacement experiments, provided there exists, for some microscopic reason, a power-law noise. At finite scales, the anomalous-growth period is not negligible. This can cause another important crossover.

IV. SCALING ARGUMENT FOR THE EDWARDS-WILKINSON CASE

We now apply the scaling argument in Sec. III to the EW model with power-law noise to reproduce most of the results in Sec. II. We have represented the power-law noise for the EW model in Eq. (10) which, when compared to Eqs. (20) and (32), corresponds to an unmodulated profile $p(y) = \delta(y)$. This representation is different from the spike function $p(y) = p^*(y)$ defined in Eq. (33) for the KPZ case, and the difference is crucial. This is made evident by examining the early development of a noise event. For the EW case, the diffusion operator $\nu\nabla^2 h$ conserves the spatial integral of $h(x,t)$. Thus the δ function is a suitable choice for representing a finite disturbance, while the spike function disappears as it evolves. For the KPZ case, it is well known that a simple parabolic peak spreads in time. Consider

$$h(x, 0) = a - x^2/b, \tag{34}$$

where a and b are constants. We have discussed in Sec. III A that Eq. (27) reduces to Eq. (19), and we can neglect other rare noise events for the early stage of the evolution. The surface evolves under only the $\frac{\lambda}{2}(\nabla h)^2$ term and the solution is

$$h(x,t) = a - x^2/(b + 2\lambda t). \tag{35}$$

A spike corresponds to $b \to 0$, and the solution describes a well-defined expanding parabola with fixed height for $t > 0$. However, the δ function corresponds to $a \to \infty$, which does not give a physical surface. The relevant quantity characterizing a peak for the KPZ case is the peak height, while that for the EW case is the peak area. Therefore, in the continuum description, the noise profiles have to be represented differently by $p^*(y)$ and $\delta(y)$, respectively, even though they are incorporated similarly in the discrete models. This difference also illustrates the nonadditive nature of the noise in the KPZ case, since two neighboring noise events corresponding to two spikes will merge to a single parabola of the height of the higher spike only, instead of the sum.

Having a representation of the noise, similar scaling arguments as in Sec. III reproduce the scaling exponents in Eq. (9) derived earlier in Sec. II. For this linear model, we can calculate the limiting properties of $\lim_{\eta_m\to 0} W(L_0, \eta_m)$ explicitly. We first express the solution of the EW model with power-law noise in Eqs. (7) and (10) as

$$h(x,t) = \sum_{\eta_i\ (\geq \eta_m)} G(x - x_i, t - t_i)\eta_i, \tag{36}$$

where G is the Green's function satisfying the boundary condition. Summing of the frequently occurring small noise events with $\eta_m \leq \eta_i \leq \eta_{m0}$ gives

$$h(x,t) = h'(x,t) + \int d^d x' dt'\ G(x - x', t - t') \times \int_{\eta_m}^{\eta_{m0}} d\eta\ n(x,t,\eta)\ \eta\,, \tag{37}$$

where the remaining terms for $\eta_i \geq \eta_{m0}$ are collected in $h'(x,t)$, which represents a realization of a surface with lower cutoff η_{m0}. The random variable $n(x,t,\eta)$ is the number density of the noise events at x and t of magnitude η with a mean $\overline{n}(\eta)$ given in Eq. (22). Taking the variance of Eq. (37), we get

$$W(L_0,\eta_m)^2 = W(L_0,\eta_{m0})^2 + \int d^dx'dt'\; G^2(x-x',t-t') \times \int_{\eta_m}^{\eta_{m0}} d\eta\; \sigma_n^2(\eta)\eta^2, \tag{38}$$

where $\sigma_n^2(\eta) = \overline{n}(\eta)$ is the variance of $n(x,t,\eta)$. The dependence of $W(L_0,\eta_m)$ on η_m comes from the integral over η, which can be evaluated using Eq. (22) and gives

$$W(L_0,\eta_m)^2 = \begin{cases} c_1 - c_2\eta_m^{2-\mu} & \text{for } \mu \neq 2 \\ c_3 - c_4 \ln(\eta_m) & \text{for } \mu = 2. \end{cases} \tag{39}$$

The c_i's are constants independent of η_m. Therefore $\lim_{\eta_m\to 0} W(L_0,\eta_m) \to 0$ for $\mu < 2$, as argued in Sec. III C. For $\mu < 2$, the limit diverges, indicating that the power-law tail fails to dominate. At $\mu = 2$, the marginal divergence leads to the same logarithmic correction in Eq. (17).

V. ANOMALOUS GROWTH

In this work, we introduced a surface growth rule in the KPZ universality class defined in Eq. (28) in order to suppress anomalous growth. This discrete model can be interpreted as ballistic deposition with partial sliding. Consider a particle being dropped to a column of height much lower than that of a neighbor. It will neither stick to the area near the top of the neighboring column as in usual ballistic-deposition models, nor slide downward all the way to the bottom of the column as in random-deposition models. Instead, it only slides downward halfway and then sticks. Adopting the present growth rule has dramatic effects on the development of large fluctuations. We simulated Zhang's model at $\mu = 2$ with this rule. A snapshot of an aggregate is shown in Fig. 4, where there is a developing large fluctuation. According to the halfway-sliding rule, a big step created by the giant particle is split in the next time step into two neighboring steps of halved heights. The splitting continues and the steps diffuse back to a smooth surface efficiently. This is in sharp contrast to the case of usual ballistic-deposition models where the big steps propagate laterally with their size decaying much more slowly [20]. Therefore, the present growth rule is able to suppress much of the anomalous growth and decrease the crossover effect discussed in Sec. III D. The very smooth parabolic noise profile $p(y)$ used in the simulations in Sec. III B, is also chosen to minimize the anomalous growth. Using other ballistic-deposition rules or profiles can lead to slightly larger values in the measurement of α. However, the qualitative features are unchanged. Similar increases in the measured α due to anomalous growth have been

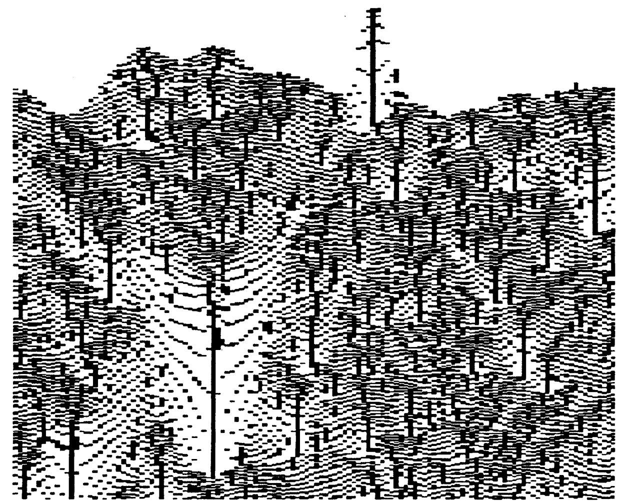

FIG. 4. Growing aggregate of ballistic-deposition model with partial sliding for power-law noise of $\mu = 2$ and $L = 150$.

explained in the context of long-range correlated noise [15].

The above precautions against anomalous growth are able to enlarge the region in the parameter space where the continuum description is valid (see Fig. 1 of Ref. [17]). However, for small μ or large noise intensity A, so that the surface is highly stretched, the dynamics would still be dominated by anomalous growth. In an earlier work, we have demonstrated that, for $\mu = d+1$, the scaling exponents α and z can be deduced by examining anomalous growth without any reference to the continuum equation [20]. We now show further that although the continuum description gives the correct α and z, it is in fact not an appropriate description at $\mu \simeq d+1$ because it gives an incorrect prediction of scaling with respect to the power-law-noise intensity. This is revealed in Fig. 3. For $1+1$ dimensions at $\mu = 2$, the scaling converges to $W \sim A^{1/2}$ at large A (thick dotted line), contradicting the continuum exponent of $\frac{1}{3}$ from Eq. (26). The correct exponent $\frac{1}{2}$ can be derived by examining the anomalous growth as follows. The substitute for the continuum evolution equation (19), which becomes invalid now, is simply that the steps advance one lattice constant laterally per unit time, irrespective of the local slope and curvature [15]. The model now characterized by Eqs. (20)–(22) and a fixed lateral propagation velocity is invariant under the transformation $A = cA'$, $h(x,t) = c^{1/2}h'(x,t)$, $\eta = c^{1/2}\eta'$, and $\eta_m = c^{1/2}\eta_m'$ from which the result follows. Previous simulations [7,8,14] are deep in this anomalous-growth regime. Therefore the crossovers characterized by the continuum description shown in Fig. 2 of Ref. [17] for $\mu \simeq d+1$ were not observed.

It has been suggested that the surfaces in Zhang's model are multi-self-affine instead of self-affine and obey

$$c_q(x,t) = \frac{1}{L}\sum_{x'=1}^{L} \mid h(x',t) - h(x'+x,t) \mid^q \sim x^{qH_q}. \tag{40}$$

If H_q depends nontrivially on q, the surface is multi-self-affine. This dependence was found in previous simulations [10]. However, in Sec. III, we have shown that the surfaces are self-affine so that $H_q \equiv \alpha$ independent of q. We suggest that the apparent multi-self-affinity results from anomalous growth. For the standard ballistic-deposition growth rule used in Ref. [10], the dynamics of the lateral propagation of big steps implies that the step height $\delta = |h(x',t) - h(x'+2,t)|$ follows a power-law distribution $1/\delta^\mu$ at large δ [20]. Therefore $c_q(x,t)$ is not a well-defined quantity for sufficiently large q, at least for small x, and diverges with increasing L. This explains the result obtained [10]. We computed $c_q(x,t)$ using the ballistic-deposition model with partial sliding where anomalous growth is much reduced. We observed a dramatic improvement in the self-affine scaling. However, even after anomalous growth is suppressed, $c_q(x,t)$ at large q can be problematic. This is because the continuum description in Sec. III and our simulations do not rule out a genuine power-law tail of small amplitude in the probability distribution of $|h(x',t) - h(x'+x,t)|$. We believe that well-defined convergent quantities will always have simple self-affine scaling behavior.

VI. DISCUSSION

We have given an exact scaling theory for a generalization of the KPZ equation with power-law noise that should describe the asymptotic behavior of Zhang's model. One may be tempted to apply the standard perturbative renormalization-group technique [2] directly. We have explained in Secs. III D and IV that the proper representation of the power-law noise is the spike function in Eq. (33). However, in the unperturbed problem with only the $\nu\nabla^2 h$ term, the spikes disappear readily. However, they give finite fluctuations because $\nu = 0$ at the fixed point when the power-law noise is relevant. As a result, a perturbative scheme with $\nu\nabla^2 h$ being the unperturbed term might not easily accommodate power-law noise.

Previous simulations on Zhang's model indicate that the exponent identity (5) is satisfied rather well [5,7,8], even though the exponents themselves deviate significantly from the true asymptotic values. The identity was checked by measuring, in addition to α, the early-time exponent β from the scaling relation $W \sim t^\beta$ for the initial growth of a flat surface. For the EW case, Eq. (18) means that this scaling also suffers from the logarithmic correction, so that β would be overestimated. This is similar for the KPZ case. Since both α and β are overestimated, the effect on the identity, which can be written as $\alpha(1+1/\beta) = 2$, is partially canceled. Therefore, it is not surprising that the exponent identity is quite robust against crossover effects.

The logarithmic correction to scaling at the transition point is quite generic. Similar behavior has been suggested and explained for bounded long-range correlated noise [16]. To arrive at a more unified picture, we look at the current problem from a slightly different point of view. After coarse graining the system by a factor of b, to enforce invariance, the lower cutoff η_m of the power-law-noise magnitude should be raised to $b^\alpha \eta_m$. The truncated power-law noise with magnitude between η_m and $b^\alpha \eta_m$ is lumped into the Gaussian noise. This is analogous to the result that renormalization of long-range correlated noise generates Gaussian noise [13]. At the transition, both the power-law and the Gaussian noise are equally relevant. Therefore, upon each stage of coarse graining, both noises would be invariant if the Gaussian component were not increasing due to the new contribution from the truncated power-law noise. In fact, the Gaussian component keeps growing and causes the logarithmic correction. Asymptotically, the ratio of the intensity of the power-law noise to the Gaussian noise approaches 0. The analogous zero ratio for the correlated noise case has been proved using the dynamical renormalization-group techniques [13].

We believe that this work offers a definitive theoretical understanding of Zhang's model. However, whether the model explains the experiments is another question. For the fluid-displacement experiments, Rubio *et al.* reported $\alpha = 0.73 \pm 0.03$ [3], while Horváth, Family, and Vicsek obtained $\alpha \simeq 0.81$ [4]. Recently, He, Kahanda, and Wong found that the effective value of α varies continuously over 0.65–0.91, depending on a capillary number [25]. Some other seemingly related recent experiments includes a bacteria-colony expansion experiment [26], an experiment of wet-front propagation in paper [27], and a paper-burning experiment [28]. The reported values of α are respectively 0.78, 0.63, and 0.7. The KPZ equation has been very successful in describing a diverse range of discrete models, including ballistic deposition, the Eden model, and directed polymers. Many of the experiments mentioned above appear to have many common features with these models. It is quite surprising that the KPZ equation, which predicts $\alpha = 0.5$, apparently does not account for any of them. An alternative unified description has not emerged. Apart from Zhang's model of power-law noise [5], models where local pinning is important have also been investigated [29,27]. Some of them might be compatible with Zhang's model if power-law noise is effectively generated from depinning. Long-range correlated noise is another candidate for some of the experiments [13]. Most of these possible explanations include some free parameters, so that the predicted values of α, or effective α, cover wide and overlapping ranges. It is evident that measurement of scaling exponents alone is not likely to resolve the current confusion in explaining the experiments.

A more thorough analysis of the dynamics of both the experimental and simulated surfaces is essential. To this end, Horváth, Family, and Vicsek have computed the tail of the noise distribution and correlation in a fluid-displacement experiment and indicated the existence of power-law noise with short-range correlation [6]. Therefore, the local pinning models can only be valid candidates for explaining the fluid-displacement experiments if power-law noise with comparable power and extent can be measured from simulations. It should be fruitful to carry out a similar analysis of the noise for other experiments or computer models.

For a complete analysis of the surface dynamics, our inverse method [30] might be the most promising approach. It is a general procedure to compute the continuum evolution equation from data. We intend to apply it to simulated and experimental surfaces to resolve some of the questions raised here.

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